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Discrete Space-time and Lorentz Symmetry

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Recent observations of Ultra High Energy Cosmic rays suggest a small violation of Lorentz symmetry. Such a violation is expected in schemes with discrete/quantized space-time. We examine this situation and suggest tests which could be carried out by, for example NASA's forthcoming GLAST Satellite.

KEY WORDS: Lorentz; symmetry; violation.

1. INTRODUCTION

Recent observations of Ultra High Energy Cosmic Rays suggest that there could be a small violation of Lorentz symmetry at energies $\sim 10^{20}$ eV, as we will see below. This is because there is the so-called Lorentz compatible GZK cut off, beyond which there should not be any particles reaching the earth from cosmological distances. This has prompted several authors including Glashow and Coleman, Mestres, Jacobson and others in recent years to speculate on the form of Lorentz violation.

Actually as pointed out by t'Hooft, the author himself and others, a high-energy violation of Lorentz symmetry is expected in schemes where space-time is discrete. Such schemes have been studied for a long time—from the work of Snyder, Finkelstein, Kardyshevskii, Wolf, the author himself and others (Snyder, 1947a,b; Finkelstine, 1996; Sidharth, 2001; Kardyshevskii, 1962; Wolf, 1990). This is encountered in Lattice Gauge Theory too (Wilson,1983), though more as a computational tool. More recently, t'Hooft and others have re-examined lattice theories. This time the motivation has been more on the lines of minimum space-time intervals (t'Hooft, 1996).

In this case is there is a departure from Lorentz Symmetry (Sidhart, 2001; Montway and Miinster, 1994). Typically we have an energy momentum relation

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(with units such that $c = 1 = \hbar$)

$$E^2 = m^2 + p^2 - l^2 p^4 \tag{1}$$

where l is a minimum length interval, which could be typically the Planck length and more generally the Compton length, (which reduces to the Planck length for a Planck mass). Interestingly we could arrive at (1) from an alternative point of view, starting directly from the noncommutativity, or the modified uncertainty principle which results from these considerations in Quantum Gravity or Quantum Superstring theory, for example (cf. (Sidharth, 2003) and references therein):

$$[x, p] = \hbar' = \hbar \left[1 + \left(\frac{l}{\hbar}\right)^2 p^2 \right] \text{etc.} \tag{2}$$

where we have temporarily re-introduced $\hbar$. (2) shows that effectively $\hbar$ is replaced by $\hbar'$. So,

$$E = (m^2 + p^2)^{\frac{1}{2}} (1 + l^2 p^2)^{-1}$$

or

$$E^2 = m^2 + p^2 - 2l^2 p^4, \tag{3}$$

neglecting higher-order terms. (3) is of the same form as (1). We now examine a few implications of (1).

2. MODIFIED DISPERSION

Let us consider an effect similar to the Compton effect (Powell and Craseman, 1961) but with (1) replacing the usual energy momentum formula. Here if $\vec{k}_0$ is the incident radiation and $\vec{k}$ is the scattered radiation at an angle Θ, as in the usual theory we get from the energy and momentum conservation laws,

$$k_0 - k = E - m \tag{4}$$

and

$$\vec{k}_0 - \vec{k} = \vec{p} \tag{5}$$

Further algebraic manipulation of (4) and (5) gives

$$kk_0(1 - cos\Theta) = m(k_0 - k) + \frac{l^2}{2}[Q^2 + 2mQ]^2 = mQ + \frac{l^2}{2}[Q^2 + 2mQ]^2$$

where

$$E - m = Q = k_0 - k$$

Whence, we get the frequency k as, (in natural units),

$$k = \frac{mk_0 + \frac{l^2}{2}[Q^2 + 2mQ]^2}{[m + k_0(1 - cos\Theta)]} \tag{6}$$

Alternatively, let us denote the additional change in frequency due to the noncommutativity of space-time or the presence of minimum space-time intervals by ϵ, so that

$$k + \epsilon = \bar{k}$$

$\bar{k}$ being the usual compton frequency. With this, we get, instead of (6),

$$\epsilon = \frac{l^2[Q^2 + 2mQ]^2}{2\{m + k_0(1 - cos\Theta)\}} \tag{7}$$

The relations (6) or (7) enable us to observe the effect of the violation of Lorentz Symmetry as embodied in (1).

It must be reiterated that in the usual formulation there is a restriction on the energy of the cosmic rays that we receive due to the presence of the GZK cut off (cf. references in (Gonzales Mestres, 1997); (Olinto, 2000)): The Microwave Cosmic Background Radiation limits the propagation of Ultra High Energy Cosmic Rays due to inelastic collisions with the background photons. Particles with energy more than a few 10^{19} eV cannot propagate more than about 50 Mpc. On the other hand, very small departure from Lorentz Symmetry as in (6) or (7) would lead to significant effects at higher and higher energies and could explain the observed Ultra High Energy Cosmic Ray events of energy greater than 10^{20} eV.

3. PARTICLE BEHAVIOUR

Owing to (1) we have a modified Klein–Gordon equation

$$(D + l^2\nabla^4 - m^2)\psi = 0 \tag{8}$$

where D denotes the usual D'Alembertian.

Just to get a feel, it would be interesting to consider the extra effect in (8). For simplicity we take the one-dimensional case. As in conventional theory if we separate the space and time parts of the wave function, we get

$$l^2u^{(4)} + u^{(2)} + \lambda u = 0, \quad \lambda = E^2 - m^2 > 0 \tag{9}$$

where $u^{(n)}$ denotes the nth space derivative.

Whence if in (9) we take,

$$u = e^{\alpha x}$$

and $\alpha^2 = \beta$ we get,

$$l^2\beta^2 + \beta + \lambda = 0$$

whence

$$\beta = \frac{-1 \pm \sqrt{1 - 4l^2\lambda}}{2l^2}$$

So

$$\beta \approx \frac{-1 \pm \{1 - 2l^2\lambda\}}{2l^2} \tag{10}$$

From (10) it is easy to deduce that there are two extra solutions, as can be anticipated by the fact that (8) is a fourth-order equation, unlike the usual second-order Klein–Gordon equation. Thus, we have

$$\beta = -\lambda (< 0)$$

giving the usual solutions, but additionally we have

$$\beta = -\left(\frac{1 - \lambda l^2}{l^2}\right)(< 0) \tag{11}$$

What do the two extra solutions in (11) indicate? To see this we observe that α is given by, from (11)

$$|\alpha| \approx \pm\frac{1}{l} \tag{12}$$

In other words (12) corresponds to waves with wavelength of the order l, which is intuitively quite reasonable.

It is interesting to note that if l is an absolute length then the extra effect is independent of the mass of the particle. In any case the solutions from (12) are GZK violating solutions, arising as they do, from the modified energy momentum formula (1).

We now make some conclusions. Departures from Lorentz symmetry of the type given in (1) have as noted, been studied, though from a phenomenological point of view (Gonzales Master, 1997; Coleman and Galshow, 1999; Jacobson, 2002; Olinto, 2000; Carroll, 2001; Nagano, 2000). These arise mostly from an observation of Ultra High Energy Cosmic Rays. Given Lorentz Symmetry, there is the GZK cut off already alluded to, such that particles above this cut off would not be able to travel cosmological distances and reach the earth. However there are indications of a violation of the GZK cut off (cf. references in (Gonzales Master, 1997)-(Nagano, 2000)).

In any case some of the effects following, for example from (1), like (6) or (7) can be detected, it is hoped by the GLAST Satellite to be launched by NASA in 2006 or shortly thereafter (http://glast.gsfc.nasa.gov/).

Interestingly, if in (1) or (8) we take, purely on an *ad hoc* basis, $-l^2$ rather than $+l^2$, we get two real exponential solution of (8). One of them is an increasing exponential leading to very high probabilities for finding these particles.

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